MATHEMATICS

1. $f(x) = \log_e(e - e^x)$

$\therefore$ for log $(e - e^x)$ to be defined $e - e^x > 0 \Rightarrow y \in (-\infty, 1)$

2. $\frac{1}{\sqrt{2}}\left(\cos\frac{7\pi}{5} - \sin\frac{2\pi}{5}\right) = \cos\frac{\pi}{4}\cos\frac{7\pi}{5} + \sin\frac{\pi}{4}\sin\frac{7\pi}{5}$

$= \cos\left(\frac{7\pi}{5} - \frac{\pi}{4}\right) = \cos\left(\pi + \frac{3\pi}{20}\right) = \cos\left(\pi - \frac{3\pi}{20}\right) = \cos\left(\frac{17\pi}{20}\right)$

3_. $3 - \{x\} = \log_2(9 - 2^{\{x\}}) \Rightarrow 2^3 . 2^{-\{x\}} = 9 - 2^{\{x\}}$

$\Rightarrow t^2 - 9t + 8 = 0 \Rightarrow t = 1, 8 \Rightarrow 2^{\{x\}} = 1, 8 \Rightarrow \{x\} = 0$

4_. Points of intersection of $y = f(x)$ and $y = f^{-1}(x)$ are $(0, 0)$ (π, π), …….

5_. $f(\theta) = \frac{\sqrt{2}\sin\theta}{\sqrt{1-2\sin^2\theta}} = \pm\sqrt{\frac{2\sin^2\theta}{1-2\sin^2\theta}}$

6_. Let $\phi(x) = xP(x) - 1 \Rightarrow \phi(x) = \lambda(x - 1)(x - 2) \ldots\ldots (x - 99)$

$\phi(0) = -\lambda(99!) \Rightarrow \lambda = \frac{1}{\lfloor 99} \Rightarrow 100P(100) - 1 = 1 \Rightarrow P(100) = \frac{1}{50}$

7_. $f(2) = 1 - 2f(1)$
$f(3) = -2 - 2f(2)$
$f(4) = 3 - 2f(3)$

$f(2010) = 2009 - 2f(2009)$
Adding all, $3[f(2) + f(3) + \ldots. + f(2009)] + f(2010) + 2f(1) = (1 + 3 + \ldots\ldots + 2009) - (2 + 4 + \ldots.. + 2008)$
$\Rightarrow 3[f(1) + f(2) + \ldots.. + f(2009)] = 1005$

8_. $f(g(x)) = x \Rightarrow f'(g(x)).g'(x) = 1 \Rightarrow g'(x) = \frac{1}{f'(g(x))} = \frac{1}{1+\sec^2(g(x))} = \frac{1}{2+\tan^2(g(x))}$

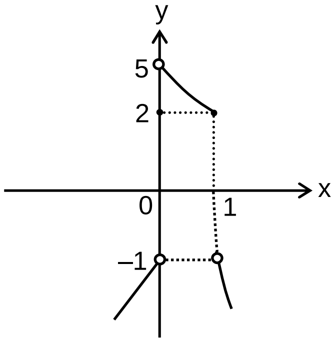
9. $\tan^{-1}\left(\frac{x}{1+\sqrt{1-x^2}}\right) + \frac{2.\sqrt{\frac{1-x}{1+x}}}{1+\frac{1-x}{1+x}} = \sqrt{1-x^2} \Rightarrow \tan^{-1}\left(\frac{x}{1+\sqrt{1-x^2}}\right) + \sqrt{1-x^2} = \sqrt{1-x^2} \Rightarrow x = 0$

10. $\frac{x^3}{2\sin^2\left(\frac{1}{2}\tan^{-1}\frac{x}{y}\right)} + \frac{y^3}{2\cos^2\left(\frac{1}{2}\tan^{-1}\frac{y}{x}\right)} = \frac{x^3}{1-\cos\left(\tan^{-1}\frac{x}{y}\right)} + \frac{y^3}{1+\cos\left(\tan^{-1}\frac{y}{x}\right)}$

$= \frac{x^3}{1-\frac{|y|}{\sqrt{x^2+y^2}}} + \frac{y^3}{1+\frac{|x|}{\sqrt{x^2+y^2}}} = (x + y)(x^2 + y^2)$

11_. $f(x) = \tan^{-1}\left(\dfrac{\sqrt{12}-2}{x^2+2+\dfrac{3}{x^2}}\right) = \tan^{-1}\left(\dfrac{2\sqrt{3}-2}{x^2+2+\dfrac{3}{x^2}}\right)$ $\left(x^2+\dfrac{3}{x^2} \geq 2\sqrt{3}\right)$

12_*. $gof\,(x) = \begin{cases} 2x-1, & x<0 \\ 2, & x=0 \\ (2-x)^2+1, & 0<x\leq 1 \\ 1-2x, & x>1 \end{cases}$



13_*. $f(x) = x$, $g(x) = |x|$, $h(x) = \dfrac{1}{x}$

14_*. $\dfrac{dy}{dx} = \dfrac{1}{(1+|x|)^2} > 0$ $\Rightarrow$ one-one

$R_f = (-1, 1)$ $\Rightarrow$ into

15_*. $\tan^{-1}(|x^2+2x| + |x+3| - ||x^2+2x| - |x-3||) = \pi - \cot^{-1}\left(-\dfrac{1}{2}\right)$

$= \pi - (\pi - \cot^{-1}\dfrac{1}{2}) = \cot^{-1}\dfrac{1}{2} = \tan^{-1} 2$

(i) $|x^2+2x| \geq |x+3|$ $\Rightarrow$ $2|x+3| = 2$ $\Rightarrow$ $x = -2, -4$ $\Rightarrow$ $x = -4$

(ii) $|x^2+2x| \leq |x+3|$ $\Rightarrow$ $2|x^2+2x| = 2$ $\Rightarrow$ $x = -1, -1 \pm \sqrt{2}$ $\Rightarrow$ $x = -1+\sqrt{2}, -1$

$\Rightarrow$ $\alpha = -4, \beta = -1, \gamma = -1+\sqrt{2}$

16_*. $h(\omega) = 0$ and $h(\omega^2) = 0$ $\Rightarrow$ $\omega f(1) + \omega^2 g(1) = 0$ and $\omega^2 f(1) + \omega g(1) = 0$ $\Rightarrow$ $f(1) = g(1) = 0$

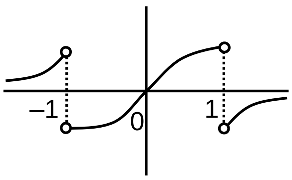
17_*. Solution possible if ordered pair $([\sin^{-1}x], [\cos^{-1}x]) = (0, 0), (1, 0), (1, 1)$

18_*. $\dfrac{x}{\sqrt{1+x^2}} = 2|x|$ $\Rightarrow$ $x = 0$ $\Rightarrow$ $a = 0$

19_*. $f(x) = 1 - \dfrac{2}{2^{\{x\}}+1}$ and $2 \leq 2^{\{x\}} + 1 < 3$

20_*. $f'(x) = -\sin x(\cos x)^{\cos x}(1 + \ln \cos x)$

21_*.



22_*. $fog(x) = \begin{cases} 1-\sqrt{x} & ; \; x \in Q \\ (1-x)^2 & ; \; x \notin Q \end{cases}$

$\Rightarrow$ $fog(\sqrt{2}-1) = fog(3-\sqrt{2})$ $\therefore$ many-one Also into

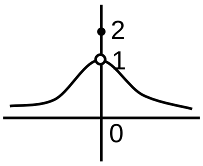
23_*. $f(-x) = f(x)$ $\Rightarrow$ $-\alpha x^3 - \beta x - \tan x.\text{sgn}(x) = \alpha x^3 + \beta x - \tan x.\text{sgn}(x)$ $\Rightarrow$ $2(\alpha x^3 + \beta x) = 0$

$\Rightarrow \quad \alpha = 0 \ \& \ \beta = 0 \quad \Rightarrow \quad [a] = 1,4 \text{ and } \{a\} = \frac{1}{2}, \frac{1}{3} \quad \Rightarrow \quad a = \frac{3}{2}, \frac{4}{3}, \frac{9}{2}, \frac{13}{3}$

24_*. $f(x) = \cot^{-1}((x + 2)^2 + \alpha^2 - 3\alpha - 4)$. For f(x) to be onto, $\alpha^2 - 3\alpha - 4 = 0$

25_*. Let $\cos^{-1}x = t \quad \Rightarrow \quad 2t = a + \frac{a^2}{t} \quad \Rightarrow \quad 2t^2 - at - a^2 = 0 \Rightarrow \quad t = a, -\frac{a}{2}$ where $t \neq 0$

26*. Let $\sin^{-1}x = \theta$; $\theta \in \left[-\frac{\pi}{6}, \frac{\pi}{6}\right] \quad \therefore \quad f(x) = \sin^{-1}(\sin 3\theta) = 3\theta = 3\sin^{-1}x$

27_*. 

at x = 0, f(0) = 2 for $x \neq 0$, $f(x) = 0 + \frac{1}{1+x^2} = \frac{1}{1+x^2}$

28_*. $f(x) = \begin{cases} 3x & : x \geq 0 \\ x & : x < 0 \end{cases}$ and $g(x) = \begin{cases} \frac{x}{3} & : x \geq 0 \\ x & : x < 0 \end{cases} \quad \therefore \quad h(x) = x$

29*. f(x) is strictly incr. function

31*. $f(x) = (x - a)^2 + a$
Now $f(x) = f^{-1}(x) \quad \Rightarrow \quad f(x) = x \Rightarrow \quad (x - a)^2 + a = x \Rightarrow \quad x - a = 0, 1 \Rightarrow x = a, a + 1$

32*. $f^n(x) = \left(\frac{3}{4}\right)^n x + \left(\frac{3}{4}\right)^{n-1} + \left(\frac{3}{4}\right)^{n-2} + \dots\dots\dots + \frac{3}{4} + 1 \quad \Rightarrow \quad \lim_{n\to\infty} f^n(x) = 0 + \frac{1}{1-\frac{3}{4}} = 4$

33_. $x^4 - 4x^2 - \log_2 y = 0 \quad \Rightarrow \quad x^2 = 2 \pm \sqrt{4 + \log_2 y}$

34_. $g(x) = 1 + \frac{6}{\sin x - 2} \in [-5, -2]$

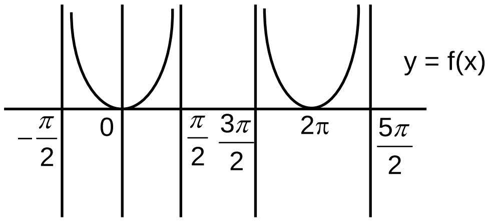
35_. $g^{-1} : [-5, -2] \to \left[\frac{\pi}{2}, \pi\right]$ and $f : [2, \infty) \to [1, \infty)$

$g^{-1}(x) \geq 2 \Rightarrow \quad x \geq g(2) \quad \Rightarrow \quad x \geq \frac{4 + \sin 2}{\sin 2 - 2} \quad \Rightarrow \quad x \in \left[\frac{4 + \sin 2}{\sin 2 - 2}, -2\right]$

36. Let $g'(1) = a \ \& \ g''(2) = b$ then $f(x) = x^2 + ax + b$
and $g(x) = (1 + a + b)x^2 + x(2x + a) + 2 = (a + b + 3)x^2 + ax + 2$
$\Rightarrow \quad g'(x) = 2(a + b + 3)x + a \ \& \ g''(x) = 2(a + b + 3)$
$\therefore \quad g'(1) = 2(a + b + 3) + a = a \ \& \ g''(2) = 2(a + b + 3) = b$
$\Rightarrow \quad a + b + 3 = 0 \ \& \ 2a + b + 6 = 0$
$\therefore \quad f(x) = x^2 - 3x$ and $g(x) = -3x + 2$

37. Area $= \int_{-\sqrt{2}}^{\sqrt{2}} \left(-3x + 2 - x^2 + 3x\right)dx = \frac{8\sqrt{2}}{3}$.

38_. $f(x) = \log(\sec x)$ $\Rightarrow$ $D_f \in \bigcup\limits_{n \in I} \left(2n\pi - \frac{\pi}{2}, 2n\pi + \frac{\pi}{2}\right)$



$g(x) = f'(x) = \tan x$

$D_g = \left(2n\pi - \frac{\pi}{2}, 2n\pi + \frac{\pi}{2}\right)$

Fundamental period of g(x) is 2π

$gog^{-1}(x) = \tan(\tan^{-1}x) = x$ for all $x \in R$

39. $g(f(x)) = x$ $\Rightarrow$ $g'(f(x))\, f'(x) = 1$. Put $f(x) = -\frac{7}{6}$ ie. $x = 1$

40. Put $x = y = 1$ $\Rightarrow$ $f^2(1) - f(1) - 6 = 0$ $\Rightarrow$ $f(1) = 3$ Now put $y = 1$ and $x = \frac{1}{2}$.